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REMARKS ON SOME PROPERTIES OF THE EQUATION OF HEAT TRANSFER IN MULTICHANNEL EXCHANGERS

T. ZALESKI and A. B. JARZEBSKI

Polish Academy of Sciences, Research Centre of Chemical Engineering and Chemical Apparatus Construction, Gliwice, Poland

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Abstract—Some properties of the equation of heat transfer in n-channel mixed parallel and counter-flow exchangers are discussed. It is shown that the matrix of coefficients **A** is of rank $n-1$, and the necessary and sufficient condition for **A** to have two zero latent roots is given.

NOMENCLATURE

- a_{ij}, number of transfer units per unit of coordinate x; $a_{ij} = k_{ij}h_{ij}/W_i$;
- **A**, coefficient matrix;
- **A**′, **B**′, **C**′, matrices of order $n-1$;
- **B**, diagonal matrix;
- **C**, $n \times n$ matrix;
- $|\mathbf{A}'|, |\mathbf{B}'|, |\mathbf{C}'|$, determinants of **A**′, **B**′ and **C**′, respectively;
- h_{ij}, common perimeter of channels i and j;
- **I**, unit matrix;
- k_{ij}, overall surface conductance for heat transfer between channels i and j;
- λ, latent root;
- n, number of channels;
- t, temperature column vector, $t = [t_1, t_2, \ldots.. t_n]^T$
- t_i, temperature of fluid in channel i;
- W, fluid heat capacity rate;
- x, space coordinate along the channel.

THE theoretical analysis of the temperature distribution in the fluids in multichannel mixed parallel and counter-flow heat exchangers, for steady state heat transfer, is described with sufficient accuracy, by a system of linear ordinary differential equations with constant coefficients

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \mathbf{A}t \qquad (1)$$

where t is the column matrix of temperatures and **A** is a square matrix of order n

$$\mathbf{A} = \begin{bmatrix} -\sum\limits_{i=2}^{n} a_{1i} & a_{12} \cdots\cdots\cdots a_{1n} \\ a_{21} & -\sum\limits_{\substack{i=1 \\ i\neq 2}}^{n} a_{2i} \cdots\cdots a_{2n} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ a_{n1} & a_{n2} \cdots\cdots - \sum\limits_{i=1}^{n-1} a_{ni} \end{bmatrix}$$

Wolf presents the general solution of system (1) assuming that multiple zero latent roots of **A** cannot exist [1]. He proves that all the latent roots are distinct with the same assumption. Settari proves that all the latent roots are real [2]; whilst not negating Wolf's theorem concerning distinct non-zero latent roots, he shows that two zero latent roots can exist, giving a plate exchanger consisting of four channels connected in series (Fig. 1) as an example.

It may be noted that multiple zero latent roots may exist in other types of exchangers as well [3–6].

An analysis of Settari's paper has led us to the following conclusions.

I. The existence of two zero latent roots leads Settari to conclude that **A** may be doubly degenerate, probably because of an incorrect interpretation of a theorem given by Hildebrand [7]. As we shall see later, this would imply a particular geometry and structure of an exchanger.

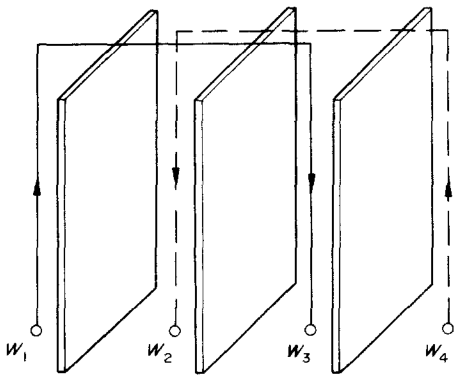


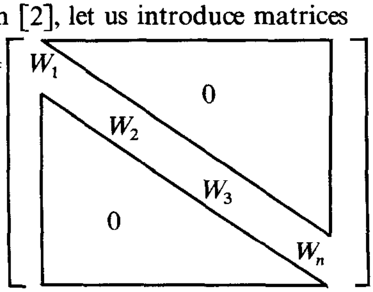
FIG. 1. Typical plate exchanger geometry.

We shall now prove

Theorem 1
If **A** is not a diagonal block matrix, the rank of **A** is always equal to $n - 1$.

Proof
As in [2], let us introduce matrices

B =



det **B** $\neq 0$

and

$$\mathbf{C} = \begin{bmatrix} c_{11}\, c_{12} \cdots\cdots & c_{1n} \\ c_{21}\, c_{22} \cdots\cdots & c_{2n} \\ \cdots\cdots\cdots & \cdots \\ c_{n1}\, c_{n2} \cdots\cdots & c_{nn} \end{bmatrix}$$

where $c_{ij} = h_{ij}k_{ij} = c_{ji}$ for $i \neq j$

$$c_{ii} = -\sum_{j=1}^{n} h_{ij}k_{ij} \text{ and } k_{ii}h_{ii} = 0$$

can be represented by the product

$$\mathbf{C} = \mathbf{B}\mathbf{A}. \tag{2}$$

Striking out the last row and column of **A**, **B** and **C** we obtain matrices **A**′, **B**′ and **C**′ respectively, of order $n - 1$. Clearly

$$\mathbf{C}' = \mathbf{B}'\mathbf{A}'. \tag{3}$$

As **C**′ is symmetric it must be irreducible. It is also diagonally dominant with strict inequality for $c_{n-1,n-1}$, then **C**′ is an irreducibly diagonal dominant matrix. Hence from theorem 1.8 ([8] p. 23) we conclude that **C**′ is non-singular and $|\mathbf{C}'| = |\mathbf{B}'||\mathbf{A}'|$ then **A**′ is non-singular and $|\mathbf{A}'|$ does not vanish completing the proof.

Diagonal block matrices are excluded from consideration because any such matrix represent two or more thermally isolated sub-exchangers.

II. The possibility of multiple zero latent roots is determined by

Theorem 2
The necessary and sufficient condition for **A** to have two zero latent roots is that the sum of diagonal elements of **B** is zero.

Proof
We first prove the following lemma
All the cofactors of **C** are equal.

Proof. Let us consider the cofactors of c_{ni} and $c_{n,i+1}$, denoted by C_{ni} and $C_{n,i+1}$ respectively

$$C_{ni} = (-1)^{n+i} \begin{vmatrix} c_{11}\, c_{12} \dots c_{1,i-1}\, c_{1,i+1} \\ \qquad c_{1,i+2} \dots c_{1n} \\ c_{21}\, c_{22} \dots c_{2,i-1}\, c_{2,i+1} \\ \qquad c_{2,i+2} \dots c_{2n} \\ \dots\dots\dots\dots \\ c_{n-1,1}\, c_{n-1,2} \cdot\cdot c_{n-1,i-1} \\ \qquad c_{n-1i+1}\, c_{n-1,1+2} \cdot\cdot c_{n-1,n} \end{vmatrix}$$

$$C_{n,i+1} = (-1)^{n+i+1} \begin{vmatrix} c_{11}\, c_{12} \dots c_{1,i-1}\, c_{1i} \\ \qquad c_{1,1+2} \dots c_{1n} \\ c_{21}\, c_{22} \dots c_{2,i-1}\, c_{2i} \\ \qquad c_{2,i+2} \dots c_{2n} \\ \dots\dots\dots\dots \\ c_{n-1,1}\, c_{n-1,2} \cdot\cdot c_{n-1,i-1} \\ \qquad c_{n-1,i} c_{n-1,i+2} \dots c_{n-1,r} \end{vmatrix}$$

The properties of **C** show that C_{ni} is obtained when we add all the other columns to column i of $C_{n,i+1}$. Hence all the cofactors of the entries of the bottom row of **C** are equal. The same method can be used to prove that the cofactors of an arbitrary i-th row are all equal. If they are denoted by C_i the cofactors of the column n are $C_1, C_2, \dots\dots C_n$ respectively, but as **C** is symmetric, they must be equal to those of the n-th row, thus

$$C_1 = C_2 = \dots\dots = C_n = C_A$$

completing the proof of the lemma.

Theorem 1 shows that $C_A \neq 0$

From (2) we may write

$$\mathbf{A} = \mathbf{B}^{-1}\mathbf{C}. \qquad (4)$$

Let $B_1^{-1}, B_2^{-1}, \dots\dots B_n^{-1}$ be the cofactors of the diagonal elements of $\mathbf{B}^{-1}$.

The sum of first principal minors of **A** is then

$$(B_1^{-1} + B_2^{-1} + \dots + B_n^{-1})\, C_A$$

but as it is the coefficient of the linear term of the characteristic equation $|\mathbf{A} - \mathbf{I}\lambda| = 0$, the necessary and sufficient condition for **A** to have two zero latent roots is

$$B_1^{-1} + B_2^{-1} + \dots + B_n^{-1} = \frac{\sum_{i=1}^{n} W_i}{\prod_{i=1}^{n} W_i} = 0 \qquad (5)$$

hence

$$\sum_{i=1}^{n} W_i = 0$$

which had to be proved.

III. The heat exchanger considered by Settari ([2], Fig. 1) and shown in Fig. 1 of this communication always fulfils the conditions of theorem 2 as $W_3 = -W_1$ and $W_2 = -W_4$. Therefore, the matrix **A** arising from this heat exchange system possesses two zero latent roots irrespective of the direction of flow in the two circuits. This result differs from that of Settari [2] who concludes that the number of zero latent roots is one or two, depending on whether or not the flows are in the same direction.

It has been shown [9] that for plate heat exchangers with number of channels divisible by four, and channels connected as in Fig. 1, the matrix has two and only two zero latent roots. This is, of course, in accordance with theorem 2.

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REMARQUES SUR QUELQUES PROPRIETES DE L'EQUATION DE TRANSFERT THERMIQUE DANS DES ECHANGEURS MULTI-CANAUX

Résumé—On discute quelques propriétés de l'équation de transfert thermique dans des échangeurs à n canaux parallèles et à contre-courant. On montre que la matrice des coefficients A est de rang $n - 1$ et on donne la condition nécessaire et suffisante pour que A ait deux valeurs propres nulles.

BEMERKUNGEN ZU EINIGEN EIGENSCHAFTEN VON WÄRMEÜBERGANGSGLEICHUNGEN FÜR VIELKANAL-WÄRMEÜBERTRAGER

Zusammenfassung—Einige Eigenschaften von Wärmeübergangsgleichungen für Vielkanal-Gleich- und Gegenstromwärmeübertrager werden diskutiert. Es wird gezeigt, dass die Matrix der Koeffizienten A vom Grad $n - 1$ ist und die für A notwendige und hinreichende Bedingung von zwei latenten Null-Wurzeln ist angegeben.

О НЕКОТОРЫХ СВОЙСТВАХ УРАВНЕНИЯ ПЕРЕНОСА ТЕПЛА В МНОГОКАНАЛЬНЫХ ТЕПЛООБМЕННИКАХ

Аннотация—Рассматриваются некоторые свойства уравнения переноса тепла в прямоточных и противоточных теплообменниках, состоящих из n каналов. Показано, что матрица коэффициентов A имеет ранг $n-1$. Приводится условие, необходимое и достаточное для того, чтобы коэффициенты A имели два нулевых характеристических корня.